VII International Conference on Computational Methods for Coupled Problems in Science and Engineering
COUPLED PROBLEMS 2017
M. Papadrakakis, E. Oñate and B. Schrefler (Eds)

ADAPTATIVE REDUCED ORDER MODEL TO CONTROL NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS

MOURAD OULGHELOU*, CYRILLE ALLERY†

* †LaSIE, UMR 7356 CNRS, Université de La Rochelle Pôle Science et Technologie
Avenue Michel Crépeau, 17042 La Rochelle Cedex 1, France
* mourad.oulghelou@univ-lr.fr , †cyrille.allery@univ-lr.fr

Key words: Model Order Reduction, POD, PGD, Grassmann manifold, interpolations, sub optimal control

Abstract. In classical adjoint based optimal control of unsteady dynamical systems, requirements of CPU time and storage memory are known to be very important. To overcome this issue, model order reduction techiques operating by the construction of a separated representation of the solution are considered. A spatial basis must be calculated for each variation in control parameters, followed by a Galerkin projection of the equations's residuals on this basis, that results in a low dimentional system of ordinary differential equations. These steps need to be carried out in every iteration of the control algorithm. The most popular reduced order model method is the Proper Orthogonal Decomposition (POD). It is used here for the construction of reduced bases. The interest in this communication is turned to the adaptation of these bases respectivly to control parameter variations. Two adaptation approaches are considered. The first one uses a powerfull interpolation method based on calculus of geodesic paths on the Grassmann manifold. This approach needs a precomputed set of bases associated to a distribution of opetating points, that are calculated using POD method. The second approach uses the Proper Generalized Decomposition (PGD) considered here as a correction method. This method consists in enriching a basis by reducing the error of the approximated solution.

1 Inroduction

In this communication, in order to reduce simulation CPU time of control problems, suboptimal control methodology is considered. It relies on reduced order models techniques allowing to represent the high fidelity solutions as a space time separated form by using for instance the POD [1]. It is expected that only few space basis elements will provide a reasonably good description of the desired dynamic. The projection of governing equations onto this reduced basis results in a set of differential equations with considerably smaller order than the degree of freedom arising from the full equations. In the previous

works established in the topic of sub-optimal control, the POD method was the mostly investigated model reduction approach. For instance Bergmann et al. [2] controlled the time angular velocity of a rotating cylinder to optimize drag of an incompressible viscous flow past a circular cylinder. Ravindran [3] considered the control problem of reducing recirculations behind the step of a flow in a backward facing step channel, where the control was effected either through the movement of a wall portion or through blowing on it. By the parameter variation, the basis was updated using the POD method trough the generation of a new set of snaphots. However this approach still costly, eventhough it is less time consuming compared to the full control. We can cite also the work of Akman [4] who controlled diffusion-convection-reaction equations using POD method to generate reduced bases. In addition, he used subspace angle interpolation method (SAIM) to adapt a basis for new configuration. It consists in adapting two availabale reduced order bases constructed for two different operating points to fit with a new operating point. Tallet et al. [5] controlled in quasi-real time the boundary condition of the anisothermal Navier-Stokes equations in a lid driven cavity. The POD basis considered is generated once for all from multiple configurations, in a manner there is no need to update the spatial basis within the control algorithm, only reduced equations were solved. In the following this approach will be refered to as global basis method (GBM). In this communication, some improvements are proposed to the aformentionned approaches. Using the tools of calculus of geodesic paths on the Grassmann manifold, the notion of subspace angle interpolations appears to be a particular case of to the Grassmann interpolation method (GIM) where more operating points are involved. Consequently, an enhancement of bases adaptation due to this enrichment of samples space is expected. This approach was already used by Amsallem et al. in the context of aeroelasticity to adapt POD reduced order bases to fit with a new physical parameter [6]. The second approach that we propose here for bases adaptation, is the Proper Generalized Decomposition (PGD) [7, 8, 9, 10]. It acts as an enrichment tool of bases within the control loop, and can be seen as a generalization of the POD method.
This communication is organized as follows. Section 2 describes the studied case full control problem. Section 3 introduces the POD reduced order model method used through this communication to construct the set of bases samples required when the control is based on interpolation methods, such as RBF, Lagrange, SAIM, and GIM. The last method is described in Section 4. Section 5 presents the PGD approach used to adapte POD bases bases. in Section 6, the reduced optimal control problem is introduced. Finally, Section 7 presents the numerical results of sub-optimal control, obtained using the aformentioned methods.

2 Optimal control problem formulation

Let $\Omega \subset \mathbb{R}^2$ a bounded domain with boundary $\Gamma \subset \mathcal{C}^2$ and $\mathcal{I} = [0, T]$. Consider the nonstationary nonlinear heat equations with initial condition $u_0 \in L^2(\Omega)$ submitted to a

body force $f_\gamma \in L^2(\mathcal{I}, H^{-1})$ described by the equations

$$\begin{cases} \dfrac{\partial}{\partial t}u - \nu\Delta u + \dfrac{1}{2}u^2 = f_\gamma \quad \text{in } \Omega \times \mathcal{I} \\ u = g \quad \text{on } \Gamma \times \mathcal{I} \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

We define the cost functional for all $\kappa > 0$

$$\mathcal{J}(u,\gamma) = \frac{1}{2}\int_0^T \int_\Omega |u - \hat{u}|^2 dxdt + \frac{1}{2}\int_\Omega |u_T - \hat{u}_T|^2 dx + \frac{\kappa}{2}|\gamma|^2. \tag{2}$$

where the last term $\frac{\kappa}{2}|\gamma|^2$ serves as a regularization of the constrained minimization problem defined by

$$\min_{\gamma \in U_{ad}} \mathcal{J}(u,\gamma) \tag{3}$$

such that u solves the equations (1) and $U_{ad} = \big\{\gamma \in \mathbb{R}^d \quad s.t. \quad |\gamma| < C\big\}$. In other words, acting on the parameter γ we want to control equations (1) in a way that the seeked parameter is a minimimizer for the cost functional (2). A descent gradient method is used for the numerical search of that minimizer. The adjoint problem is then derived

$$\begin{cases} \dfrac{\partial}{\partial t}\xi + \nu\Delta\xi - u\xi = u - \hat{u} \quad \text{in } \Omega \times \mathcal{I} \\ \xi = 0 \quad \text{on } \Gamma \times \mathcal{I} \\ \xi(T) = \hat{u}(T) - u(T) \quad \text{in } \Omega. \end{cases} \tag{4}$$

and the functional gradient writes

$$\nabla\mathcal{J}(\gamma) = -\int_0^T \int_\Omega \nabla f_\gamma \xi dxdt + \kappa\gamma \tag{5}$$

3 POD reduced order model method

In this section, we seek an approximate separated form of u such as

$$u \approx u_m = \sum_{j=1}^{m} \alpha_j \Phi_j \tag{6}$$

Using the POD method, the optimal separated representation of the solution is achieved in just few basis elements. This basis is extracted from the information contained in a set of realizations called snapshots [1]. Let $(u^1, \ldots, u^M)$ be an ensemble of snapshots of the considered system of evolution respectivly to time instances $(t_1, \ldots, t_M)$. The POD basis is calculated in four steps

(i) build the correlation matrix K from the considered snapshots whose elements are given as $K_{ij} = \frac{1}{M}\int_\Omega u^i u^j dx$ for $1 \leq i, j \leq M$

(ii) compute the eigenvalues $\lambda_1 > \cdots > \lambda_M$ and eigenvectors $y^1, \ldots, y^M$ of K

(iii) set $\Phi_i = \sum_{j=0}^{M} y_j^i u^j$ for $1 \leq i \leq M$

(iv) normalize $\Phi_i = \frac{\Phi_i}{||\Phi_i||}$ for $1 \leq i \leq M$

In order to obtain a low dimentional basis, the modes corresponding to small eigenvalues are neglected. Only the m more energetic first modes are considered ($m << M$). The high fidelity equations are then projected onto this basis, resulting in system of ordinary differential equations of low order (ROM) given by

$$\begin{cases} \sum_{j=1}^{m} M_{ij}\frac{d}{dt}\alpha_j + \nu\sum_{j=1}^{m} R_{ij}\alpha_j + \frac{\nu}{\epsilon}\sum_{j=1}^{m} B_{ij}\alpha_j + \frac{1}{2}\sum_{j,k=1}^{m} N_{ijk}\alpha_j\alpha_k = F_i + \frac{\nu}{\epsilon}G_i \\ \sum_{j=1}^{m} M_{ij}\alpha_j(0) = \langle u_0, \Phi_i\rangle_\Omega \\ \forall i \in 1, \cdots, m \end{cases} \tag{7}$$

$$\begin{aligned} M_{ij} &= \langle \Phi_j, \Phi_i\rangle_\Omega \qquad & N_{ijk} &= \langle \Phi_j\Phi_k, \Phi_i\rangle_\Omega \qquad & F_i &= \langle f, \Phi_i\rangle_\Omega \\ R_{ij} &= \langle \nabla\Phi_j, \nabla\Phi_i\rangle_\Omega \qquad & B_{ij} &= \langle \Phi_j, \Phi_i\rangle_\Gamma \qquad & G_i &= \langle g, \Phi_i\rangle_\Gamma \end{aligned} \tag{8}$$

where $\langle u, v\rangle_\Omega = \int_\Omega uvdx$ and $\langle u, v\rangle_\Gamma = \int_\Gamma uvd\sigma$. The essential boundary condition is enforced in a weak integral form using the penalty method [11]. It consists in writing u in the boundary as

$$u_{/\Gamma} = g - \epsilon\frac{\partial u}{\partial n} \tag{9}$$

where ϵ is a small parameter. So that we can replace the essential boundary condition by the alternative natural boundary condition $\frac{\partial u}{\partial n} = \frac{u_{/\Gamma} - g}{\epsilon}$. This remark explains the presence of the elements B_{ij} and G_i in the ROM equations (7). As $\epsilon \to 0$ the domain boundary will converge to the original boundary condition.

4 ROMs interpolation

An important and practical issue associated with bases produced by the POD method, is their lack of robustness with respect to parameter variations. An attractive idea for adapting them is to use methods of interpolation. Unfortunately, the standard interpolation such as RBF or Lagrange methods of a set of bases does not necessarily produce a basis. However, there exists an appropriate interpolation technique called here the Grassmann interpolation method (GIM) that can always ensure this property.
Let's define first the tangent space at a point of the Grassmann manifold and the concept of Geodesic path between two points on a differential manifold. Consider n_g orthogonal bases of the same dimension generated from the set of configuration parameters $\vartheta = \{\gamma_0, \gamma_1, \ldots, \gamma_{n_g-1}\}$. An important result about the Grassmann manifold $\mathcal{G}(N_b, N_f)$ states that at each of its points $\mathcal{S}$ there exists a tangent space denoted $\mathcal{T}_\mathcal{S}$ [12] of the same dimension, and with an origin the point of tangency. Each point of this tangent space $\mathcal{T}_\mathcal{S}$ can be as well represented by a matrix $\Gamma \in \mathbb{R}^{N_f \times N_b}$. The tangent space is a vector space. Hence, $\mathcal{T}_\mathcal{S}$ is a flat space in which interpolations can be performed as usual.

Let ψ denote an orthogonal matrix whose columns span $\mathcal{S} \in \mathcal{G}(N_b, N_f)$ and χ denote a point of $\mathcal{T}_S$ spanned by the columns of Γ. The exponential mapping Exp maps χ to an N_b dimensional subspace $\mathcal{S}'$ represented by an orthogonal matrix $\psi' \in \mathbb{R}^{N_f \times N_b}$ and based on thin singular value decomposition

$$\begin{cases} \Gamma = U\Sigma V^T \\ \psi' = \psi V \cos(\Sigma) + U \sin(\Sigma) \end{cases} \tag{10}$$

The link between the tangent space $\mathcal{T}_\mathcal{S}$ at a point $\mathcal{S}$ of the Grassmann manifold and the manifold itself is established by the exponential mapping, and explicitly expressed by equations (10). The inverse of this map is defined between a neighborhood of $\mathcal{S}$ and $\mathcal{T}_\mathcal{S}$ by the logarithmic map. This mapping is denoted by $\log_s$ and defined in a neighborhood of $\mathcal{S} \in \mathcal{G}(N_b, N_f)$.
The following result enables the practical computation of the logarithmic map. Let ψ and ψ' denote two orthogonal matrices whose columns span a subspace $\mathcal{S}$ and $\mathcal{S}'$ in the neighborhood of S, respectivly. The image of $\mathcal{S}'$ by the logarithmic map $\log_S$, $\chi = \log_s(\mathcal{S}') \in \mathcal{T}_S$, is represented by the matrix Γ given by

$$(I - \psi\psi^T)\psi'(\psi^T\psi')^{-1} = U\Sigma V^T$$

$$\Gamma = U \tan^{-1}(\Sigma) V^T$$

Let $\{\psi_i \in \mathbb{R}^{N_f \times N_b}\}_{i=0}^{n_g-1}$ denote the representing matrices of $\{\mathcal{S}_i\}_{i=0}^{n_g-1}$ the set of N_b-dimensional subspaces of $\mathbb{R}^{N_f}$ associated with a set $\{\gamma\}_{i=0}^{n_g-1}$ of control parameters. Then the basis adaptation algorithm is resumed in four steps as described in the paper of Amsallem and Farhat [6]

GIM bases interpolation steps:

- **step 0.** Chose a reference point $\mathcal{S}_{i_0}$ to be the origine point of the interpolation.
- **step 1.** using the logarithm map, map each $\mathcal{S}_i$ to a matrix Γ_i representing a point χ_i of $\mathcal{T}_{\mathcal{S}_{i_0}}$

$$(I - \psi_{i_0}\psi_{i_0}^T)\psi_i(\psi_{i_0}^T\psi_i)^{-1} = U_i\Sigma_i V_i^T \qquad \Gamma_i = U_i \tan^{-1}(\Sigma_i)V_i^T \tag{11}$$

- **step 2.** interpolate Γ_{n_b} associated to the target control parameter γ_{n_b}
- **step 3.** using the exponential map $\mathrm{Exp}_{\mathcal{S}_{i_0}}$, map Γ_{n_b} to a subspace $\mathcal{S}_{n_b}$ on $\mathcal{G}(N_b, N_f)$ spanned by a matrix ψ_{n_b}

$$\Gamma_{n_b} = U_{n_b}\Sigma_{n_b}V_{n_b}^T \qquad \psi_{n_b} = \psi_{i_0}V_{n_b}\cos(\Sigma_{n_b}) + U_{n_b}\sin(\Sigma_{n_b}) \tag{12}$$

5 PGD reduced order model method

Consider now a POD basis associated to a parameter $\gamma \in \mathbb{R}^2$ which is insufficient to achieve a good approximation of the solution associated to a neighbor parameter. Assume that a solution u is approximated in this basis like $u_m = \sum_{i=0}^{m}\alpha_i\Phi_i$. The goal is to enrich this expression by a couple $(\alpha_{m+1}, \Phi_{m+1})$ such that $u_{m+1} = u_m + \alpha_{m+1}\Phi_{m+1}$ represents well the solution u.
This can be achieved using the Proper Generalized (PGD) method [7, 8, 9, 10]. Let's start by explaining how to determine the new couple $(\alpha_{m+1}, \Phi_{m+1})$. Suppose that α_{m+1} is known and fixed, $\Phi_{m+1} = \mathcal{S}(\alpha_{m+1})$ is defined in

$$V_{m+1} = \left\{ \Phi_{m+1} \in H^1(\Omega)/\Phi_{m+1_{/\Gamma}} = \frac{1}{\langle \alpha_{m+1}, \alpha_{m+1}\rangle_{\mathcal{I}}} \left(g - \sum_{j=1}^{m} \langle \alpha_j, \alpha_{m+1}\rangle_{\mathcal{I}} \Phi_{j_{/\Gamma}} \right) \right\}$$

for all $\psi \in H_0^1(\Omega)$ by the folowing Galerkin orthogonality criteria

$$\begin{aligned} \sum_{j=1}^{m+1} \left\langle \frac{d}{dt}\alpha_j, \alpha_{m+1} \right\rangle_{\mathcal{I}} \langle \Phi_j, \psi\rangle_\Omega + \nu \sum_{j=1}^{m+1} \langle \alpha_j, \alpha_{m+1}\rangle_{\mathcal{I}} \langle \nabla\Phi_j, \nabla\psi\rangle_\Omega \\ + \frac{1}{2}\sum_{j=1}^{m+1} \langle \alpha_j\alpha_k, \alpha_{m+1}\rangle_{\mathcal{I}} \langle \Phi_j\Phi_k, \psi\rangle_\Omega = \langle \alpha_{m+1} f, \psi\rangle_{\Omega\times\mathcal{I}} \end{aligned} \tag{13}$$

Similarly, if Φ_{m+1} is known and fixed, we define $\alpha_{m+1} = \mathcal{T}(\Phi_{m+1})$ for all $\beta \in L^2(\mathcal{I})$

$$\begin{aligned} \sum_{j=1}^{m+1} \langle \Phi_j, \Phi_{m+1}\rangle_\Omega \left\langle \frac{d}{dt}\alpha_j, \beta \right\rangle_{\mathcal{I}} + \nu \sum_{j=1}^{m+1} \langle \nabla\Phi_j, \nabla\Phi_{m+1}\rangle_\Omega \langle \alpha_j, \beta\rangle_{\mathcal{I}} + \frac{\nu}{\epsilon}\sum_{j=1}^{m+1} \langle \Phi_j, \Phi_{m+1}\rangle_\Gamma \langle \alpha_j, \beta\rangle_{\mathcal{I}} \\ + \frac{1}{2}\sum_{j=1}^{m+1} \langle \Phi_j\Phi_k, \Phi_{m+1}\rangle_\Omega \langle \alpha_j\alpha_k, \beta\rangle_{\mathcal{I}} = \langle \beta f, \Phi_{m+1}\rangle_{\Omega\times\mathcal{I}} + \frac{\nu}{\epsilon}\langle \beta g, \Phi_{m+1}\rangle_{\Gamma\times\mathcal{I}} \end{aligned} \tag{14}$$

Hence, a natural definition of an optimal couple $(\alpha_{m+1}, \Phi_{m+1})$ relies in satisfying simultaniously equation (13) and (14). The problem then reads

$$\text{Find } (\alpha_{m+1}, \Phi_{m+1}) \in \mathcal{I} \times V_{m+1} \text{ such as } \begin{cases} \alpha_{m+1} = \mathcal{T}(\Phi_{m+1}) \\ \Phi_{m+1} = \mathcal{S}(\alpha_{m+1}) \end{cases} \tag{15}$$

The problem can be formulated in terms of α_{m+1} as follow

$$\text{Find } \alpha_{m+1} \in \mathcal{I} \quad \text{such as} \quad \alpha_{m+1} = \mathcal{T} \circ \mathcal{S}(\alpha_{m+1}) \tag{16}$$

This allows to interpret equations (16) as a fixed point problem. Consequently, a classical fixed point algorithm is used to find the new couple $(\alpha_{m+1}, \Phi_{m+1})$

while $m \leq M_{max}$ **do**
  α^0_{m+1} given arbitrary;
  for $k \leftarrow 0$ **to** k_{max} **do**
    solve $\Phi^k_{m+1} = \mathcal{S}(\alpha^k_{m+1})$;
    solve $\alpha^k_{m+1} = \mathcal{T}(\Phi^k_{m+1})$;
    Check convergence $|\sigma^k_m - \sigma^{k-1}_m| < \epsilon_u |\sigma^k_m|$ where $\sigma^k_m = ||\alpha^k_{m+1}||$;
  end
  Enrichement: $\Phi^{m+1} = \text{span}\{\Phi_j\}_{j=0,\cdots,m+1}$
  Update: solve equations (7), $\Lambda^{m+1} = \text{span}\{\alpha_j\}_{j=0,\cdots,m+1}$
  Reduce basis length if necessary.
end

Algorithm 1: PGD Algorithm with update.

The basis length reduction in algorithm (1) consists in performing a POD on the snapshots formed by the reduced basis. In practice we consider only one enrichment, so $M_{max} = 1$.

6 Reduced optimal control problem

Let's now define the control problem in a reduced form, such that the only quantity being manipulated is the temporal basis obtained as a solution of the reduced order model. Let $\hat{u}$ be given in the separated form $\hat{u} = \sum_{j=1}^{\hat{m}} \hat{\alpha_j} \hat{\Phi}_m$. The reduced cost functional writes

$$\mathcal{J}(\alpha, \gamma) = \frac{1}{2} \sum_{k=1}^{m} \left(\int_0^T \alpha_k^2 dt + \alpha_k^2(T) \right) + \frac{1}{2} \sum_{l=1}^{\hat{m}} \left(\int_0^T \hat{\alpha}_l^2 dt + \hat{\alpha}_l^2(T) \right) - \sum_{k=1}^{m} \sum_{l=1}^{\hat{m}} C_{kl} \left(\int_0^T \alpha_k \hat{\alpha}_l dt + \alpha_k(T) \hat{\alpha}_l(T) \right) + \frac{\kappa}{2} |\gamma|^2. \tag{17}$$

where α is now the state variable, $C_{kl} = \left\langle \Phi_k, \hat{\Phi}_l \right\rangle_\Omega$ and $\hat{m}$ is the target solution modes number. The associated adjoint ROM equations follow immediatly from the ROM state equations (7)

$$\begin{cases} \sum_{i=1}^{m} M_{iq} \frac{d}{dt}\beta_i - \nu \sum_{i=1}^{m} R_{iq}\beta_i - \frac{\nu}{\epsilon} \sum_{i=1}^{m} B_{iq}\beta_i - \sum_{i=1}^{m} N_{iqk}\alpha_j\beta_i = \alpha_q - \sum_{i=1}^{\hat{m}} C_{qi}\hat{\alpha}_i \\ \sum_{i=1}^{m} M_{iq}\beta_q(T) = \sum_{i=1}^{\hat{m}} C_{qi}\hat{\alpha}_i(T) - \alpha_q(T) \\ \forall q = 1, \cdots, m \end{cases} \tag{18}$$

Finaly the cost functional gradient writes

$$\nabla \mathcal{J}(\gamma) = -\sum_{j=1}^{m} \int_0^T \beta_j \int_\Omega \nabla f_\gamma \Phi_j dx dt + \kappa\gamma \tag{19}$$

The reduced control algorithm is summerized as follow

γ^0 given arbitrary;
while $\mathcal{J} \leq \epsilon$ **do**
  $\Phi^0 = \Phi(\gamma^k)$ fixed;
  for $l = 0$ **to** k_{max} **do**
    solve ROM state equations (7);
    solve ROM state equations (18);
    Compute the descent direction $d_l = -\nabla \mathcal{J}(\gamma^l)$ using Eq. (19);
    Determine the new control $\gamma_l = \gamma_{l-1} + \sigma_l d_l$ ($\sigma_l > 0$ Armijo line search);
    Evaluate $\mathcal{J}(\gamma^k)$;
  end
  Update Φ^k using one of the methods (PGD, GIM, SAIM ...);
  $k = k + 1$;
end

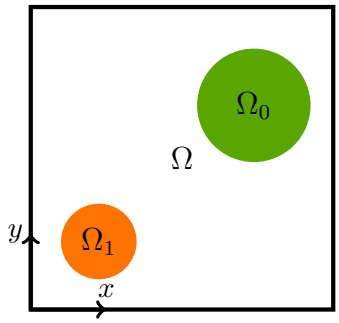
Algorithm 2: Reduced Optimal Control algorithm

7 Numerical analysis

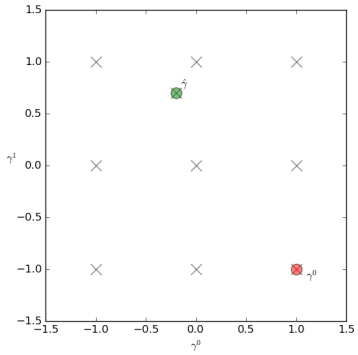
We consider the non linear heat equations in a 2D square domain $\Omega = [0,1] \times [0,1]$ with a diffusion coefficient $\nu = 10^{-2}$. The optimal control is performed by acting on the parameter $\gamma \in \mathbb{R}^2$ appearing on the source term f_γ. The velocity on the boundary is given by $u_{/\Gamma} = cos(t)(x+y)$ and the source term is defined by $f_\gamma(t,x) = \sum_{k=0}^{1} f_\gamma^{(k)}$. Where $\gamma = (\gamma^{(0)}, \gamma^{(1)}) \in \mathbb{R}^2$ and $f_\gamma^{(k)} = \beta_k(t,\gamma)\chi_{\Omega_k}(x) f_k(x)$, where $k = 0, 1$. χ is the mask

function and :

$$\begin{cases} \beta_0(t,\gamma) = \gamma^{(0)} e^{-t} & f_0(x) = x^2 + y^2 \\ \beta_1(t,\gamma) = \gamma^{(1)} sin(t) & f_1(x) = 1 \end{cases}$$



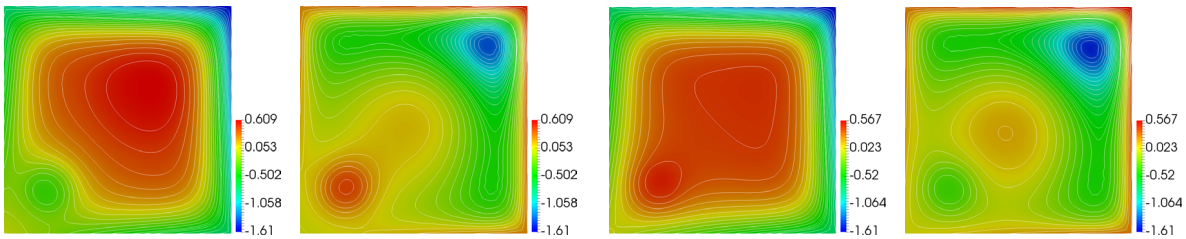
(a) Poistion of subdomains Ω_0 and Ω_1 where external forces $f_\gamma^{(0)}$ (respectively) $f_\gamma^{(1)}$ are applied.



(b) sampling distribution of control parameters, $\hat{\gamma}$ is the target parameter and γ_0 is the first guess.

Figure 1: Space domain Ω and sampling distribution of control parameters.

The high fidelity equations are solved in the time interval $\mathcal{I} = [0, 5]$ using a time step $\delta t = 0.01$ and piecewise second order continuous finite elements on a nonuniform mesh with 17728 triangles. A preconditionned GMRES method is used in the optimization steps. The target solution $\hat{u}$ is the solution associated to $\hat{\gamma} = (-0.2, 0.7)$ and the control starting guess solution is associated to $\gamma_0 = (1, -1)$. These solutions are depicted respectively in figure (2b) and figure (2a) in two time instances $t = T/2$ and $t = T$. These figures show a strongly different dynamic between the initial guess state and the target state.



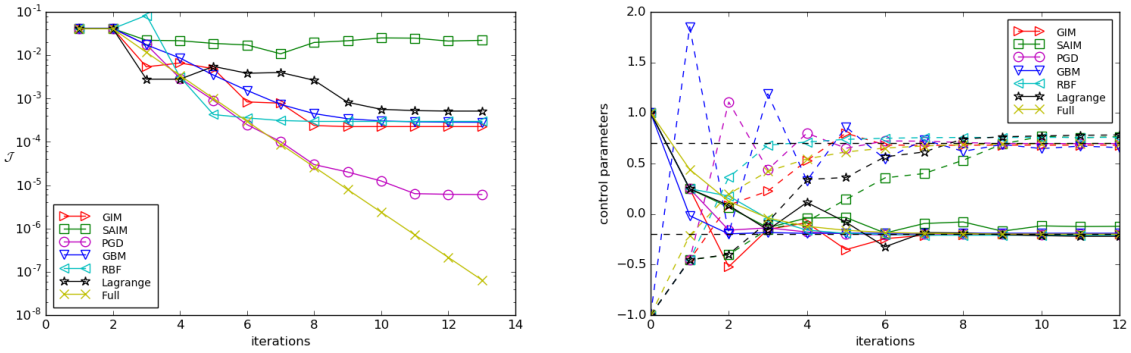
(a) first guess at $t = T/2$(left) and at $t = T$(right)

(b) Target at $t = T/2$(left) and at $t = T$(right)

Figure 2: Isovalues of the control first guess and the target solutions.

In the following, some numerical results illustrating the performance of reduced optimal control using the earlier discussed methods are presented.

The samples bases used for the interpolation methods (SAIM, GIM, RBF, Lagrange) here are genereated using the POD method over the set of operating points $\{\gamma_k\}_{k=1,\cdots,9} \in \mathbb{R}^2$ (see figure (1b)). Each basis is calculated using 500 snapshots and only 5 modes are considered. On the other hand, the snapshots associated to the set of operating points are token altogether to generate the basis used in GBM method.



(a) Functional decay

(b) Parameters evolution

Figure 3: Convergence behavior described by the functional decay (left) and the control parameter values respectivly to number of control algorithm iterations (right).

The evolution of the functional $\mathcal{J}$ according to the control algorithm iterations is presented in figure (3a). It is observed that all the methods converge with different order of accuracy. The PGD is the most accurate one with a functional that attained 3×10^{-5} in 8 iterations. Followed by the GIM and GBM methods for which the fonctional attained 2.2×10^{-4} and 2.8×10^{-4} in 9 and 12 iterations respectivly. Then by the classical interpolation methods RBF and Lagrange with a functional that attained 2.9×10^{-4} and 5.1×10^{-4} in 8 and 12 iterations respectivly. And finally the SAIM method where the functional stagnated at 2.1×10^{-2} from the itertation 14 of the control algorithm.

These results are confirmed by figure (3b) where the parameters γ respectivly to control iteration are plotted for each method. The target parameter $\hat{\gamma} = (\hat{\gamma}^{(0)}, \hat{\gamma}^{(1)}) = (-0.2, 0.7)$ corresponds to the horizontal lines plotted in figure (3b). All the methods converge approximatly towards the target parameter $\hat{\gamma}$.

Table (1) illustrates the gains in CPU time and the percenatge of errors between the target solution and the solution achieved using GIM, SAIM, GBM, RBF, Lagrange, and PGD methods within the control algorithm. this error is defined by ε and expressed by

$$\varepsilon = 100 \times \int_0^T \frac{||\hat{u} - u||_{L^2(\Omega)}}{||\hat{u}||_{L^2(\Omega)}} dt$$

The algorithm for each method is stopped when a stagnation of the corresponding functional occurs. τ designate the time needed for the classical adjoint based control algo-

rithm to attain 0.11% of error in 11 iterations. using the interpolation methods GIM, RBF, Lagrange as well as the Global basis method GBM, the reduced optimal control is approximatly about 80 times faster with errors of 1.15%, 8.12%, 11.22% and 6.92% respectivly. the same gain is achieved by the SAIM method, except that the error in this case is of order about 72%. It is worth noting that for the GIM method, further gain in accuracy versus an acceptable loss in CPU time is expected if this method and the PGD method are appropriatly combined.

method	ε	time	γ
Full control	0.11%	τ	$(-0.198, 0.695)$
PGD	1.15%	$\tau/10$	$(-0.206, 0.697)$
GIM	5.88%	$\tau/84$	$(-0.199, 0.681)$
GBM	6.92%	$\tau/80$	$(-0.190, 0.647)$
RBF	8.12%	$\tau/78$	$(-0.209, 0.758)$
Lagrange	11.22%	$\tau/81$	$(-0.212, 0.772)$
SAIM	72.31%	$\tau/88$	$(-0.117, 0.767)$

Table 1: Gains in time obtained using optimal reduced control relativly to the full adjoint based control at iteration 11.

8 Conclusion and Discussion

The aim of this communication was to illustrate the potential gain that can be offered by the use of the GIM and PGD methods in optimal control of non linear partial differential equations. The results of control using PGD method showed a very good accuracy (errors of 1.15%). The most shining property of this method is that it does not need any sampling bases near a parameter to adapt a given basis. The approaches using interpolations such as GIM, RBF and Lagrange as well as the global basis approach GBM give satisfaying results (errors of 5.88%, 8.12%, 11.22% and 6.92% respectivly) with an important speed up in comparison with the full control problem. The main inconvenient of these approaches is that they need a set of precomputed data in different operating points of the full model problem. The generation of a good sampling of these data may need an important offline CPU time of calulations. More particulary, for some problem cases where the control parameter $\gamma \in \mathbb{R}^d$ such that $d > 3$, this approach becomes expensive, and may be considered as limited.

For all these methods, the basis is updated at each iteration of the control algorithm. The CPU time can however be reduced further if the basis update is perfomed only if necessary, which is when the basis runs out of its trust region.

REFERENCES

[1] L. Sirovich, "Turbulence and the dynamics of coherent structures: I, ii and iii," *Quarterly of Applied Mathematics*, pp. 461–590, 1987.

[2] M. B. L. Cordier and J.-P. Brancher, "Optimal rotary control of the cylinder wake using proper orthogonal decomposition reduced-order model," *AIP Physics of Fluids*, 2005.

[3] S. S. Ravindran, "A reduced-order approach for optimal control of fluids using proper orthogonal decomposition," *International Journal for Nnumerical Methods in Fluids*, pp. 425–448, 2000.

[4] T. Akman, "Local improvements to reduced order approximations of optimal control problems governed by diffusion convection reaction equation," *Computers and Mathematics with Applications*, vol. 70, pp. 104–131, 2015.

[5] A. Tallet, C. Allery, and C. Leblond, "Optimal flow control using a pod based reduced-order model," *Numerical Heat Transfer, Part B*, vol. 170, 2016.

[6] D. Amsallem and C. Farhat, "An interpolation method for adapting reduced-order models and application to aeroelasticity," *AIAA Journal*, pp. 1803–1813, 2008.

[7] A. Ammar, B. Mokdad, F. Chinesta, and R. Keunings, "A new family of solvers for some classes of multidimensional partial differential equations encountered in kinetic theory modeling of complex fluids," *Journal of Non-Newtonian Fluid Mechanics*, vol. 139, no. 3, pp. 153 – 176, 2006.

[8] A. Dumon, C. Allery, and A. Ammar, "Proper general decomposition (pgd) for the resolution of navierstokes equations," *Journal of Computational Physics*, vol. 230, pp. 1387 – 1407, 2011.

[9] M. Bergmann and L. Cordier, "generalized spectral decomposition technique to solve a class of linear stochastic partial di erential equations.," *Computer Methods in Applied Mechanics and Engineering, Elsevier*, vol. 196 (45-48), pp. 521–4537, 2007.

[10] P. Ladeveze, *Nonlinear computational structural mechanics: New approaches and non-incremental methods of calculation*. Springer (New York), 1999.

[11] E. Liberge and A. Hamdouni, "Reduced order modelling method via proper orthogonal decomposition (pod) for flow around an oscillating cylinder," *Journal of Fluids and Structures*, vol. 26, no. 2, pp. 292 – 311, 2010.

[12] A. A. M. Sepulchre, "Riemann geometry of grassmann manifolds with a view on algorithmic computation," *Acta Applicandae Mathematicae*, vol. 80, Issue 2, p. 199220, 2004.